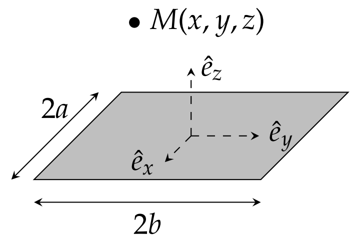
Electric field created by a charged finite plate

We want to compute analytically the electric field generated by a finite charged plate with a charge density σ at the surface. The geometry is the following:



The expression of the electric field from the Coulomb law is:

$$\vec{E}(x,y,z) = \frac{1}{4\pi\varepsilon_0}\int_{u=-a}^{a}\int_{v=-b}^{b}\frac{\sigma}{\left((x-u)^2+(y-v)^2+z^2\right)^{3/2}}\left((x-u)\hat{e}_x+(y-v)\hat{e}_y+z\hat{e}_z\right)\mathrm{d}u\mathrm{d}v$$

We start with the component E_x:

$$E_x(x,y,z) = \frac{\sigma}{4\pi\varepsilon_0}\int_{u=-a}^{a}\int_{v=-b}^{b}\frac{(x-u)\,\mathrm{d}u\mathrm{d}v}{\left((x-u)^2+(y-v)^2+z^2\right)^{3/2}}$$

The integration over x is straightforward:

$$\begin{aligned}E_x(x,y,z) &= \frac{\sigma}{4\pi\varepsilon_0}\int_{v=-b}^{b}\left[\frac{-1}{\left((x-u)^2+(y-v)^2+z^2\right)^{1/2}}\right]_{u=-a}^{a}\mathrm{d}v\\ &= \frac{\sigma}{4\pi\varepsilon_0}\int_{v=-b}^{b}\left[\frac{1}{\left((x+a)^2+(y-v)^2+z^2\right)^{1/2}}-\frac{1}{\left((x-a)^2+(y-v)^2+z^2\right)^{1/2}}\right]\mathrm{d}v\end{aligned}$$

The derivative of the function argsh(x) is $\frac{1}{\sqrt{x^2+1}}$, which tells us how to perform the integration. We introduce $c_\pm = \sqrt{(x\pm a)^2+z^2}$ and $p = \frac{y-v}{c_+}$, $q = \frac{y-v}{c_-}$ and we obtain the result for E_x. Note that there is no minus sign because the invert the limits and $c_+\mathrm{d}p = -\mathrm{d}u, c_-\mathrm{d}q = -\mathrm{d}v$.

$$\begin{aligned}E_x(x,y,z) &= \frac{\sigma}{4\pi\varepsilon_0}\left[\int_{p=\frac{y-b}{c_+}}^{p=\frac{y+b}{c_+}}\frac{c_+\mathrm{d}p}{c_+\sqrt{p^2+1}}-\int_{q=\frac{y-b}{c_-}}^{q=\frac{y+b}{c_-}}\frac{c_-\mathrm{d}q}{c_-\sqrt{q^2+1}}\right]\\ &= \frac{\sigma}{4\pi\varepsilon_0}\left[\mathrm{argsh}(\frac{y+b}{c_+})-\mathrm{argsh}(\frac{y-b}{c_+})-\mathrm{argsh}(\frac{y+b}{c_-})+\mathrm{argsh}(\frac{y-b}{c_-})\right]\\ E_x(x,y,z) &= \frac{\sigma}{4\pi\varepsilon_0}\left[\mathrm{argsh}(\frac{y+b}{\sqrt{(x+a)^2+z^2}})-\mathrm{argsh}(\frac{y-b}{\sqrt{(x+a)^2+z^2}})\right.\\ &\qquad\left.-\mathrm{argsh}(\frac{y+b}{\sqrt{(x-a)^2+z^2}})+\mathrm{argsh}(\frac{y-b}{\sqrt{(x-a)^2+z^2}})\right]\end{aligned}$$

Similarly, we get E_y:

$$E_y(x,y,z) = \frac{\sigma}{4\pi\varepsilon_0}\left[\text{argsh}(\frac{x+a}{\sqrt{(y+b)^2+z^2}}) - \text{argsh}(\frac{x-a}{\sqrt{(y+b)^2+z^2}}) - \text{argsh}(\frac{x+a}{\sqrt{(y-b)^2+z^2}}) + \text{argsh}(\frac{x-a}{\sqrt{(y-b)^2+z^2}})\right]$$

The calculation of E_z is different:

$$E_z(x,y,z) = \frac{\sigma}{4\pi\varepsilon_0}\int_{u=-a}^{a}\int_{v=-b}^{b}\frac{z\,\mathrm{d}u\mathrm{d}v}{\left((x-u)^2+(y-v)^2+z^2\right)^{3/2}}$$

We start with $X = x-u$ and $Y = y-v$. Note again that there is no minus sign because the invert the limits and $\mathrm{d}X = -\mathrm{d}u, \mathrm{d}Y = -\mathrm{d}v$.

$$E_z(x,y,z) = \frac{\sigma}{4\pi\varepsilon_0}\int_{X=x-a}^{x+a}\int_{Y=y-b}^{y+b}\frac{z\,\mathrm{d}X\mathrm{d}Y}{\left(X^2+Y^2+z^2\right)^{3/2}}$$

Now we introduce θ such as $Y = \sqrt{X^2+z^2}\tan\theta$. We have $\mathrm{d}Y = \sqrt{X^2+z^2}\frac{\mathrm{d}\theta}{\cos^2\theta}$ and $1+\tan^2\theta = \frac{1}{\cos^2\theta}$. We get:

$$\begin{aligned}
E_z(x,y,z) &= \frac{\sigma z}{4\pi\varepsilon_0}\int_{X=x-a}^{x+a}\int_{\theta=\arctan\left(y-b/\sqrt{X^2+z^2}\right)}^{\arctan\left(y+b/\sqrt{X^2+z^2}\right)}\frac{\sqrt{X^2+z^2}\,\mathrm{d}\theta}{\cos^2\theta\,\left((X^2+z^2)(1+\tan^2\theta)\right)^{3/2}}\mathrm{d}X \\
&= \frac{\sigma z}{4\pi\varepsilon_0}\int_{X=x-a}^{x+a}\frac{\mathrm{d}X}{X^2+z^2}\int_{\theta=\arctan\left(y-b/\sqrt{X^2+z^2}\right)}^{\arctan\left(y+b/\sqrt{X^2+z^2}\right)}\cos\theta\,\mathrm{d}\theta \\
&= \frac{\sigma z}{4\pi\varepsilon_0}\int_{X=x-a}^{x+a}\frac{\mathrm{d}X}{X^2+z^2}\left[\sin\left(\arctan\left(\frac{y+b}{\sqrt{X^2+z^2}}\right)\right) - \sin\left(\arctan\left(\frac{y-b}{\sqrt{X^2+z^2}}\right)\right)\right]
\end{aligned}$$

Since $\arctan\psi = \arcsin\left(\frac{\psi}{\sqrt{1+\psi^2}}\right)$, we obtain:

$$\begin{aligned}
E_z(x,y,z) &= \frac{\sigma z}{4\pi\varepsilon_0}\int_{X=x-a}^{x+a}\frac{\mathrm{d}X}{X^2+z^2}\left[\frac{\frac{y+b}{\sqrt{X^2+z^2}}}{\sqrt{1+\frac{(y+b)^2}{X^2+z^2}}} - \frac{\frac{y-b}{\sqrt{X^2+z^2}}}{\sqrt{1+\frac{(y-b)^2}{X^2+z^2}}}\right] \\
&= \frac{\sigma z}{4\pi\varepsilon_0}\int_{X=x-a}^{x+a}\frac{\mathrm{d}X}{X^2+z^2}\left[\frac{y+b}{\sqrt{(y+b)^2+X^2+z^2}} - \frac{y-b}{\sqrt{(y-b)^2+X^2+z^2}}\right]
\end{aligned}$$

We introduce θ and φ such as $X = \sqrt{(y+b)^2+z^2}\tan\theta$, and thus $\mathrm{d}X = \sqrt{(y+b)^2+z^2}\frac{\mathrm{d}\theta}{\cos^2\theta}$ in the first part and $X = \sqrt{(y-b)^2+z^2}\tan\varphi$, and thus $\mathrm{d}X = \sqrt{(y-b)^2+z^2}\frac{\mathrm{d}\varphi}{\cos^2\varphi}$ in the second part. We obtain:

$$\begin{aligned}
E_z(x,y,z) = \frac{\sigma z}{4\pi\varepsilon_0}\Bigg[&\int_{\arctan\left(x-a/\sqrt{(y+b)^2+z^2}\right)}^{\arctan\left(x+a/\sqrt{(y+b)^2+z^2}\right)}\frac{(y+b)\sqrt{(y+b)^2+z^2}\,\mathrm{d}\theta}{\left(((y+b)^2+z^2)\tan^2\theta+z^2\right)\cos^2\theta\sqrt{((y+b)^2+z^2)(1+\tan^2\theta)}} \\
&- \int_{\arctan\left(x-a/\sqrt{(y-b)^2+z^2}\right)}^{\arctan\left(x+a/\sqrt{(y-b)^2+z^2}\right)}\frac{(y-b)\sqrt{(y-b)^2+z^2}\,\mathrm{d}\varphi}{\left(((y-b)^2+z^2)\tan^2\varphi+z^2\right)\cos^2\varphi\sqrt{((y-b)^2+z^2)(1+\tan^2\varphi)}}\Bigg]
\end{aligned}$$

After cleaning up everything, we have:

$$E_z(x,y,z) = \frac{\sigma z}{4\pi\varepsilon_0}\left[\int_{\arctan\left(x-a/\sqrt{(y+b)^2+z^2}\right)}^{\arctan\left(x+a/\sqrt{(y+b)^2+z^2}\right)} \frac{(y+b)\,\mathrm{d}\theta}{(((y+b)^2+z^2)\tan^2\theta+z^2)\cos\theta} - \int_{\arctan\left(x-a/\sqrt{(y-b)^2+z^2}\right)}^{\arctan\left(x+a/\sqrt{(y-b)^2+z^2}\right)} \frac{(y-b)\,\mathrm{d}\varphi}{(((y-b)^2+z^2)\tan^2\varphi+z^2)\cos\varphi}\right]$$

Since $((y+b)^2+z^2)\tan^2\theta+z^2 = \frac{(y+b)^2\sin^2\theta+z^2}{\cos^2\theta}$, we can rewrite it as:

$$E_z(x,y,z) = \frac{\sigma z}{4\pi\varepsilon_0}\left[\int_{\arctan\left(x-a/\sqrt{(y+b)^2+z^2}\right)}^{\arctan\left(x+a/\sqrt{(y+b)^2+z^2}\right)} \frac{(y+b)\cos\theta\mathrm{d}\theta}{(y+b)^2\sin^2\theta+z^2} - \int_{\arctan\left(x-a/\sqrt{(y-b)^2+z^2}\right)}^{\arctan\left(x+a/\sqrt{(y-b)^2+z^2}\right)} \frac{(y-b)\cos\varphi\mathrm{d}\varphi}{(y-b)^2\sin^2\varphi+z^2}\right]$$

We introduce:

$$U = \frac{(y+b)\sin\theta}{z},\quad \mathrm{d}U = \frac{y+b}{z}\cos\theta\mathrm{d}\theta$$
$$V = \frac{(y-b)\sin\varphi}{z},\quad \mathrm{d}V = \frac{y-b}{z}\cos\varphi\mathrm{d}\varphi$$

The boundaries of the integrals are then defined by:

$$\begin{aligned} f(x\pm a, y\pm b) &= \frac{y\pm b}{z}\sin\left(\arctan\left(\frac{x\pm a}{\sqrt{(y\pm b)^2+z^2}}\right)\right) \\ &= \frac{y\pm b}{z}\frac{x\pm a}{\sqrt{(y\pm b)^2+z^2+(x\pm a)^2}} \end{aligned}$$

When we insert this in the expression of E_z, we obtain:

$$\begin{aligned} E_z(x,y,z) &= \frac{\sigma z}{4\pi\varepsilon_0}\left[\int_{U=f(x-a,y+b)}^{f(x+a,y+b)}(y+b)\frac{z}{y+b}\frac{1}{z^2}\frac{\mathrm{d}U}{U^2+1} - \int_{V=f(x-a,y-b)}^{f(x+a,y-b)}(y-b)\frac{z}{y-b}\frac{1}{z^2}\frac{\mathrm{d}V}{V^2+1}\right] \\ &= \frac{\sigma}{4\pi\varepsilon_0}\left[[\arctan U]_{U=f(x-a,y+b)}^{f(x+a,y+b)} - [\arctan V]_{V=f(x-a,y-b)}^{f(x+a,y-b)}\right] \\ &= \frac{\sigma}{4\pi\varepsilon_0}\left[\arctan\left(\frac{y+b}{z}\frac{x+a}{\sqrt{(y+b)^2+z^2+(x+a)^2}}\right) - \arctan\left(\frac{y+b}{z}\frac{x-a}{\sqrt{(y+b)^2+z^2+(x-a)^2}}\right)\right. \\ &\left. - \arctan\left(\frac{y-b}{z}\frac{x+a}{\sqrt{(y-b)^2+z^2+(x+a)^2}}\right) + \arctan\left(\frac{y-b}{z}\frac{x-a}{\sqrt{(y-b)^2+z^2+(x-a)^2}}\right)\right] \end{aligned}$$

To summarize:

$$E_x(x,y,z) = \frac{\sigma}{4\pi\varepsilon_0}\left[\operatorname{argsh}(\frac{y+b}{\sqrt{(x+a)^2+z^2}}) - \operatorname{argsh}(\frac{y-b}{\sqrt{(x+a)^2+z^2}}) - \operatorname{argsh}(\frac{y+b}{\sqrt{(x-a)^2+z^2}}) + \operatorname{argsh}(\frac{y-b}{\sqrt{(x-a)^2+z^2}})\right]$$

$$E_y(x,y,z) = \frac{\sigma}{4\pi\varepsilon_0}\left[\operatorname{argsh}(\frac{x+a}{\sqrt{(y+b)^2+z^2}}) - \operatorname{argsh}(\frac{x-a}{\sqrt{(y+b)^2+z^2}}) - \operatorname{argsh}(\frac{x+a}{\sqrt{(y-b)^2+z^2}}) + \operatorname{argsh}(\frac{x-a}{\sqrt{(y-b)^2+z^2}})\right]$$

$$E_z(x,y,z) = \frac{\sigma}{4\pi\varepsilon_0}\left[\arctan\left(\frac{y+b}{z}\frac{x+a}{\sqrt{(y+b)^2+z^2+(x+a)^2}}\right) - \arctan\left(\frac{y+b}{z}\frac{x-a}{\sqrt{(y+b)^2+z^2+(x-a)^2}}\right) - \arctan\left(\frac{y-b}{z}\frac{x+a}{\sqrt{(y-b)^2+z^2+(x+a)^2}}\right) + \arctan\left(\frac{y-b}{z}\frac{x-a}{\sqrt{(y-b)^2+z^2+(x-a)^2}}\right)\right]$$