Supplemental Material
Quantitative study of the response of a single NV defect in diamond to magnetic noise

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I. DERIVATION OF EQ. (9) AND EQ. (10)

In order to derive Eq. (9), it is first necessary to write Eq. (8) in terms of T_1, starting from Eq. (6). With $T_1 = 1/3\Gamma_1$, we have:

$$\mathcal{R}_{\text{cw}} = \mathcal{R}_o^\infty \frac{\mathcal{P}}{\mathcal{P}+\mathcal{P}_{\text{sat}}} \frac{1+2\beta+3T_1\Gamma_{\text{p}}^\infty \frac{\mathcal{P}}{\mathcal{P}+\mathcal{P}_{\text{sat}}}}{3+3T_1\Gamma_{\text{p}}^\infty \frac{\mathcal{P}}{\mathcal{P}+\mathcal{P}_{\text{sat}}}}$$

We introduce $X = \dfrac{\mathcal{P}}{\mathcal{P}+\mathcal{P}_{\text{sat}}}$ and compute $\dfrac{\partial \mathcal{R}_{\text{cw}}}{\partial T_1}$:

$$\frac{\partial \mathcal{R}_{\text{cw}}}{\partial T_1} = \mathcal{R}_o^\infty X \; \frac{3\Gamma_{\text{p}}^\infty X(3+3T_1\Gamma_{\text{p}}^\infty X) - 3\Gamma_{\text{p}}^\infty X(1+2\beta+3T_1\Gamma_{\text{p}}^\infty X)}{(3+3T_1\Gamma_{\text{p}}^\infty X)^2}$$

$$\frac{\partial \mathcal{R}_{\text{cw}}}{\partial T_1} = \frac{2(1-\beta)\; \mathcal{R}_o^\infty \Gamma_{\text{p}}^\infty X^2}{3(1+T_1\Gamma_{\text{p}}^\infty X)^2}$$

This leads to the following formulation of Eq. (8):

$$\text{SNR} = \delta T_1 \sqrt{\mathcal{R}_o^\infty \Delta t}\; \frac{2\Gamma_{\text{p}}^\infty (1-\beta)}{\sqrt{3}} \; \left(\frac{X}{1+T_1\Gamma_{\text{p}}^\infty X}\right)^{3/2} (1+2\beta+3T_1\Gamma_{\text{p}}^\infty X)^{-1/2}$$

To obtain Eq. (9), we need to look for the value of X which is maximizing SNR, so we compute $\dfrac{\partial \text{SNR}}{\partial X}$:

$$\frac{\partial \text{SNR}}{\partial X} = \delta T_1 \sqrt{\mathcal{R}_o^\infty \Delta t} \frac{X^{1/2}}{(1+T_1\Gamma_{\text{p}}^\infty X)^{3/2}} \frac{\Gamma_{\text{p}}^\infty \sqrt{3}(1-\beta)}{(1+2\beta+3T_1\Gamma_{\text{p}}^\infty X)^{1/2}} \left[\frac{1}{1+T_1\Gamma_{\text{p}}^\infty X} - \frac{T_1\Gamma_{\text{p}}^\infty X}{1+2\beta+3T_1\Gamma_{\text{p}}^\infty X}\right]$$

The value of $X_{\text{opt}} > 0$ verifying $\dfrac{\partial \text{SNR}}{\partial X} = 0$ is the positive solution of:

$$\frac{1}{1+T_1\Gamma_{\text{p}}^\infty X} = \frac{T_1\Gamma_{\text{p}}^\infty X}{1+2\beta+3T_1\Gamma_{\text{p}}^\infty X}$$

which is a 2$^{\text{nd}}$ order polynomial equation. This leads to Eq. (9):

$$X_{\text{opt}} = \frac{\mathcal{P}_{\text{opt}}}{\mathcal{P}_{\text{opt}}+\mathcal{P}_{\text{sat}}} = \frac{1+\sqrt{2(1+\beta)}}{T_1\Gamma_{\text{p}}^\infty}$$

Going further, in order to derive the sensitivity η_{cw} (given in Eq. (10) of the manuscript), we inject Eq. (9) into the expression of SNR obtained earlier to get SNR_{opt}:

$$\text{SNR}_{\text{opt}} = \delta T_1 \Gamma_{\text{p}}^\infty \sqrt{\mathcal{R}_o^\infty \Delta t}\; \frac{2(1-\beta)}{\sqrt{3}} \; \left(\frac{\frac{1+\sqrt{2(1+\beta)}}{T_1\Gamma_{\text{p}}^\infty}}{2+\sqrt{2(1+\beta)}}\right)^{3/2} (4+2\beta+3\sqrt{2(1+\beta)})^{-1/2}$$

$$\text{SNR}_{\text{opt}} = \delta T_1 \Gamma_{\text{p}}^\infty \sqrt{\mathcal{R}_o^\infty \Delta t}\; \frac{2(1-\beta)}{\sqrt{3}} \; \left(\frac{1+\alpha}{(2+\alpha)T_1\Gamma_{\text{p}}^\infty}\right)^{3/2} (4+2\beta+3\alpha)^{-1/2}$$

where $\alpha = \sqrt{2(1+\beta)}$.

The sensitivity $\eta_{\rm cw}$ is the minimal variation $\delta S_{B_\perp}^{\rm min}$ that can be detected, corresponding therefore to $\mathrm{SNR}_{\rm opt} = 1$. $\delta S_{B_\perp}$ is related to δT_1 by Eq. (2) such that:

$$\delta S_{B_\perp} = \frac{1}{3\gamma^2}\frac{\delta T_1}{T_1^2}$$

We obtain $\delta T_1^{\rm min}$ from $\mathrm{SNR}_{\rm opt} = 1$ evaluated in $T_1 = T_1^0$:

$$\delta T_1^{\rm min} = 3\frac{T_1^0\sqrt{T_1^0}}{\sqrt{\Delta t}}\frac{\sqrt{4+2\beta+3\alpha}}{2\sqrt{3}(1-\beta)}\left(\frac{2+\alpha}{1+\alpha}\right)^{3/2}\sqrt{\frac{\Gamma_{\rm p}^\infty}{\mathcal{R}_o^\infty}} = 3\frac{T_1^0\sqrt{T_1^0}}{\sqrt{\Delta t}}\theta_{\rm cw}$$

It results directly that (Eq. (10)):

$$\eta_{\rm cw} = \delta S_{B_\perp}^{\rm min}\sqrt{\Delta t} = \frac{\theta_{\rm cw}}{\gamma^2\sqrt{T_1^0}}$$

II. DERIVATION OF EQ. (13)

Concerning the derivation of Eq. (13), we have to start from Eq. (12). The associated variation δN of the number N of photons measured in Δt is therefore:

$$\delta N = \frac{T_L\mathcal{R}_0^\infty\Delta t}{2}\frac{2(1-\beta)}{3}\frac{\delta T_1}{T_1^2}e^{-\tau/T_1}$$

and the corresponding signal to noise ratio at $\tau = \tau_{\rm opt} = T_1^0/2$ and $T_1 = T_1^0$ is:

$$\mathrm{SNR} = \frac{\delta N}{\sqrt{\mathcal{R}_{\rm pulse}\Delta t}} = \sqrt{T_L\mathcal{R}_0^\infty\Delta t}\frac{(1-\beta)}{\sqrt{3e(1+2\beta)}}\frac{\delta T_1}{T_1^0\sqrt{T_1^0}}$$

where we assumed that $2(1-\beta)/\sqrt{e(1+2\beta)} \ll 1$. The sensitivity $\eta_{\rm pulse}$ is again obtained by setting $\mathrm{SNR} = 1$ to find the corresponding $\delta T_1^{\rm min} = 3\gamma^2(T_1^0)^2\delta S_{B_\perp}$.

$$\delta T_1^{\rm min} = \sqrt{\frac{3e(1+2\beta)}{(1-\beta)^2}}\frac{T_1^0\sqrt{T_1^0}}{\sqrt{T_L R_0^\infty\Delta t}}$$

and finally:

$$\eta_{\rm pulse} = \delta S_{B_\perp}\sqrt{\Delta t} = \sqrt{\frac{e(1+2\beta)}{3(1-\beta)^2 T_L\mathcal{R}_0^\infty}}\frac{1}{\gamma^2\sqrt{T_1^0}}$$