Supplemental Material
Quantitative imaging of exotic antiferromagnetic spin cycloids in $BiFeO_3$ thin films

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I. DETAILED CALCULATION OF THE STRAY FIELD PRODUCED BY THE SPIN DENSITY WAVE TIED TO THE CYCLOID

We want to compute the stray field $\vec{B}$ produced by a magnetic texture described by $\vec{M}(\vec{r})$. We introduce the potential Φ such that $\vec{B} = -\vec{\nabla}\Phi$. This potential can be expressed as:

$$\Phi(\vec{r}) = -\frac{\mu_0}{4\pi}\int \vec{M}(\vec{r'})\cdot\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r'}|}\mathrm{d}\vec{r'} \tag{1}$$

Eq.(1) is a convolution, so we can easily compute Φ in Fourier space:

$$\Phi = \mathcal{F}^{-1}\left[\mathcal{F}[\vec{M}]\;\mathcal{F}\left[\vec{\nabla}\left(\frac{1}{r}\right)\right]\right] \tag{2}$$

Rewriting Eq.(3) of the main text, we have, for a single $BiFeO_3$ layer of thickness a:

$$\Phi = -\frac{\mu_0}{4\pi}\int_{z'=-a/2}^{z'=a/2}\left(\iint_{-\infty}^{\infty}\left[iq_x\mathcal{F}(M_x)\mathcal{F}\left(\frac{1}{r}\right)+iq_y\mathcal{F}(M_y)\mathcal{F}\left(\frac{1}{r}\right)\right.\right.$$
$$\left.\left.-|q|\mathcal{F}(M_z)\mathcal{F}\left(\frac{1}{r}\right)\right]e^{ixq_x+iyq_y}\,\mathrm{d}q_x\,\mathrm{d}q_y\right)\mathrm{d}z' \tag{3}$$

$$\Phi = -\frac{\mu_0}{4\pi}\int_{z'=-a/2}^{z'=a/2}(I+II+III)\,\mathrm{d}z' \tag{4}$$

We will then evaluate I, II and III separetely, using :

$$\mathcal{F}\left[\frac{1}{r}\right](q) = 2\pi\frac{e^{-|q|(z-z')}}{|q|} \quad \text{for} \quad |q|\neq 0 \quad \text{and} \quad z'<z. \tag{5}$$

A. Type I cycloid, propagation along $\vec{k}_1$

For the cycloid type I propagating along $\vec{k}_1$, we describe the spin density wave $\vec{M}_{\mathrm{SW}}(\vec{r})$ with Eq.(1) of the main text:

$$\vec{M}_{\mathrm{SW}}(\vec{r}) = \frac{m_{\mathrm{DM}}}{\sqrt{6}\ a^3} \cos\left(\frac{k}{\sqrt{2}}(x-y)\right)(\hat{e}_x + \hat{e}_y - 2\hat{e}_z) \tag{6}$$

We have:

$$\begin{aligned}\mathcal{F}\left[\cos\left(\frac{k}{\sqrt{2}}(x-y)\right)\right](q) = \frac{1}{2}\,[\delta\left(\frac{k}{\sqrt{2}}+q_x\right)\delta\left(\frac{k}{\sqrt{2}}-q_y\right) \\ +\,\delta\left(\frac{k}{\sqrt{2}}-q_x\right)\delta\left(\frac{k}{\sqrt{2}}+q_y\right)\Big]\end{aligned} \tag{7}$$

Then we perform the computation of I, II and III, using Eqs.(5) and (7):

$$\begin{aligned}
I &= \iint_{-\infty}^{\infty} iq_x\ \mathcal{F}(M_x)\ \mathcal{F}\left(\frac{1}{r}\right)\ e^{ixq_x}e^{iyq_y}\,\mathrm{d}q_x\,\mathrm{d}q_y \\
I &= \iint_{-\infty}^{\infty} iq_x \frac{m_{\mathrm{DM}}}{\sqrt{6}}\ \frac{1}{2}\left[\delta\left(\frac{k}{\sqrt{2}}+q_x\right)\delta\left(\frac{k}{\sqrt{2}}-q_y\right)\right. \\
&\qquad\qquad \left. +\,\delta\left(\frac{k}{\sqrt{2}}-q_x\right)\delta\left(\frac{k}{\sqrt{2}}+q_y\right)\right] 2\pi\frac{e^{-|q|(z-z')}}{|q|}\ e^{ixq_x}e^{iyq_y}\,\mathrm{d}q_x\,\mathrm{d}q_y \\
I &= -2\pi\frac{m_{\mathrm{DM}}}{\sqrt{6}}\frac{e^{-k(z-z')}}{k}\frac{1}{2i}\left[-\frac{k}{\sqrt{2}}e^{-ikx/\sqrt{2}}e^{iky/\sqrt{2}} + \frac{k}{\sqrt{2}}e^{ikx/\sqrt{2}}e^{-iky/\sqrt{2}}\right] \\
I &= -\pi\frac{m_{\mathrm{DM}}}{\sqrt{3}}e^{-k(z-z')}\sin\left(\frac{k}{\sqrt{2}}(x-y)\right)
\end{aligned} \tag{8}$$

$$\begin{aligned}
II &= \iint_{-\infty}^{\infty} iq_y\ \mathcal{F}(M_y)\ \mathcal{F}\left(\frac{1}{r}\right)\ e^{ixq_x}e^{iyq_y}\,\mathrm{d}q_x\,\mathrm{d}q_y \\
II &= \iint_{-\infty}^{\infty} iq_y \frac{m_{\mathrm{DM}}}{\sqrt{6}}\ \frac{1}{2}\left[\delta\left(\frac{k}{\sqrt{2}}+q_x\right)\delta\left(\frac{k}{\sqrt{2}}-q_y\right)\right. \\
&\qquad\qquad \left. +\,\delta\left(\frac{k}{\sqrt{2}}-q_x\right)\delta\left(\frac{k}{\sqrt{2}}+q_y\right)\right] 2\pi\frac{e^{-|q|(z-z')}}{|q|}\ e^{ixq_x}e^{iyq_y}\,\mathrm{d}q_x\,\mathrm{d}q_y \\
II &= -2\pi\frac{m_{\mathrm{DM}}}{\sqrt{6}}\frac{e^{-k(z-z')}}{k}\frac{1}{2i}\left[+\frac{k}{\sqrt{2}}e^{-ikx/\sqrt{2}}e^{iky/\sqrt{2}} - \frac{k}{\sqrt{2}}e^{ikx/\sqrt{2}}e^{-iky/\sqrt{2}}\right] \\
II &= \pi\frac{m_{\mathrm{DM}}}{\sqrt{3}}e^{-k(z-z')}\sin\left(\frac{k}{\sqrt{2}}(x-y)\right)
\end{aligned} \tag{9}$$

$$
\begin{aligned}
III &= \iint_{-\infty}^{\infty} -|q| \mathcal{F}(M_z) \mathcal{F}\left(\frac{1}{r}\right)\ e^{ixq_x} e^{iyq_y}\, \mathrm{d}q_x\, \mathrm{d}q_y \\
III &= \iint_{-\infty}^{\infty} -|q| \frac{2m_{\mathrm{DM}}}{\sqrt{6}}\ \frac{1}{2} \left[\delta\left(\frac{k}{\sqrt{2}} + q_x\right) \delta\left(\frac{k}{\sqrt{2}} - q_y\right) \right. \\
&\qquad\qquad \left. + \ \delta\left(\frac{k}{\sqrt{2}} - q_x\right) \delta\left(\frac{k}{\sqrt{2}} + q_y\right) \right] 2\pi \frac{e^{-|q|(z-z')}}{|q|}\ e^{ixq_x} e^{iyq_y}\, \mathrm{d}q_x\, \mathrm{d}q_y \\
III &= -2\pi \frac{m_{\mathrm{DM}}}{\sqrt{6}} e^{-k(z-z')} \left[e^{-ikx/\sqrt{2}} e^{iky/\sqrt{2}} + e^{ikx/\sqrt{2}} e^{-iky/\sqrt{2}} \right] \\
III &= -4\pi \frac{m_{\mathrm{DM}}}{\sqrt{6}} e^{-k(z-z')} \cos\left(\frac{k}{\sqrt{2}}(x-y)\right) \qquad (10)
\end{aligned}
$$

We bring everything together, I and II cancel and we have:

$$
\Phi = -\frac{\mu_0}{4\pi\ a^3} \int_{z'=-a/2}^{z'=a/2} -4\pi e^{-k(z-z')} \frac{m_{\mathrm{DM}}}{\sqrt{6}} \cos\left(\frac{k}{\sqrt{2}}(x-y)\right) \mathrm{d}z' \qquad (11)
$$

Integrating over z', we obtain Eq.(4) of the main text:

$$
\boxed{\Phi = \frac{2\mu_0 m_{\mathrm{DM}}}{\sqrt{6}\ a^3} \frac{e^{-kz}}{k} \sinh\left(\frac{ka}{2}\right) \cos\left(\frac{k}{\sqrt{2}}(x-y)\right)} \qquad (12)
$$

Since $B_x = -\frac{\partial \Phi}{\partial x}$, $B_y = -\frac{\partial \Phi}{\partial y}$ and $B_z = -\frac{\partial \Phi}{\partial z}$, the stray field produced by a single $BiFeO_3$ layer is:

$$
\begin{cases}
B_x^{\text{one layer}} = \dfrac{\mu_0\ m_{\mathrm{DM}}}{\sqrt{3}\ a^3} e^{-kz} \sinh\left(\dfrac{ka}{2}\right) \sin\left(\dfrac{k}{\sqrt{2}}(x-y)\right) \\
B_y^{\text{one layer}} = -\dfrac{\mu_0\ m_{\mathrm{DM}}}{\sqrt{3}\ a^3} e^{-kz} \sinh\left(\dfrac{ka}{2}\right) \sin\left(\dfrac{k}{\sqrt{2}}(x-y)\right) \\
B_z^{\text{one layer}} = \sqrt{\dfrac{2}{3}} \dfrac{\mu_0\ m_{\mathrm{DM}}}{a^3} e^{-kz} \sinh\left(\dfrac{ka}{2}\right) \cos\left(\dfrac{k}{\sqrt{2}}(x-y)\right)
\end{cases} \qquad (13)
$$

To get the value of the field for the whole film constituted of N layers ($t = aN$), we have to sum these terms with $z \to z + ja$ ($j \in [0, N-1]$). At $z = d_{\mathrm{NV}}$, this leads to Eq.(5) and (6) of the main text:

$$
\boxed{\begin{cases}
B_x(x,y) = \mathcal{A}\ \sin\left(\dfrac{k}{\sqrt{2}}(x-y)\right) \\
B_y(x,y) = -\mathcal{A}\ \sin\left(\dfrac{k}{\sqrt{2}}(x-y)\right) \\
B_z(x,y) = \sqrt{2}\ \mathcal{A}\ \cos\left(\dfrac{k}{\sqrt{2}}(x-y)\right)
\end{cases}} \qquad (14)
$$

with

$$
\mathcal{A} = \frac{\mu_0\ m_{\mathrm{DM}}}{\sqrt{3}\ a^3} e^{-kd_{\mathrm{NV}}} \frac{1 - e^{-kt}}{1 - e^{-ka}} \sinh\left(\frac{ka}{2}\right) \qquad (15)
$$

B. Type II cycloid, propagation along $\vec{k'}_3$

1. *Perfect case*

For the cycloid type II, propagating along $\vec{k'}_3$ which is parallel to the $[11\bar{2}]$ direction, the spin density lays in the surface plane:

$$\vec{M}_{\mathrm{SW}}(\vec{r}) = \frac{m_{\mathrm{DM}}}{\sqrt{2}\ a^3} \cos\left(\frac{k}{\sqrt{6}}(x+y-2z)\right)(-\hat{e}_x+\hat{e}_y) \tag{16}$$

In this case, **no stray field is produced**. We can see it directly from the calculation:

$$I = \iint_{-\infty}^{\infty} iq_x \mathcal{F}(M_x)\ \mathcal{F}\left(\frac{1}{r}\right)\ e^{ixq_x} e^{iyq_y}\, \mathrm{d}q_x\, \mathrm{d}q_y$$

$$II = \iint_{-\infty}^{\infty} iq_y \mathcal{F}(M_y)\ \mathcal{F}\left(\frac{1}{r}\right)\ e^{ixq_x} e^{iyq_y}\, \mathrm{d}q_x\, \mathrm{d}q_y$$

$$II = -I(x \leftrightarrow y) = -I$$

$$III = 0$$

And finally $I + II + III = 0$.

2. *With a small deviation*

Experimentally, we detect some stray field from the type 2 cycloid with a propagation vector almost along $\vec{k'_3}$, but it looks like the projection of the wavevector is not completely aligned with the projection of $\vec{P}$. Therefore, we redo the calculation with $\vec{k'_3}(\alpha)$, a wavevector rotated from a small angle α compared to $\vec{k'_3} = \vec{k'_3}(0)$, with the constraint that it stays in the plane perpendicular to $\vec{P}$.

$$\vec{k'_3}(\alpha) = \frac{2\pi}{\lambda\sqrt{6}}\left[(\cos\alpha - \sqrt{3}\sin\alpha)\ \hat{e}_x + (\cos\alpha + \sqrt{3}\sin\alpha)\ \hat{e}_y - 2\cos\alpha\ \hat{e}_z\right] \tag{17}$$

This leads to the following expression of the spin density wave (Eq.(9) in the main text):

$$\begin{cases} M_x = -\dfrac{m_{\mathrm{DM}}}{\sqrt{6}\ a^3}(\sqrt{3}\cos\alpha + \sin\alpha)\ \cos\left(\vec{k'_3}(\alpha)\cdot\vec{r}\right) \\ M_y = \dfrac{m_{\mathrm{DM}}}{\sqrt{6}\ a^3}(\sqrt{3}\cos\alpha - \sin\alpha)\ \cos\left(\vec{k'_3}(\alpha)\cdot\vec{r}\right) \\ M_z = \dfrac{2m_{\mathrm{DM}}\sin\alpha}{\sqrt{6}\ a^3}\ \cos\left(\vec{k'_3}(\alpha)\cdot\vec{r}\right) \end{cases} \tag{18}$$

To go further, we need the expression of $\mathcal{F}\left[\cos\left(\vec{k_3'}(\alpha)\cdot\vec{r}\right)\right]$:

$$\begin{aligned}
\mathcal{F}\left[\cos\left(\vec{k_3'}(\alpha)\cdot\vec{r}\right)\right] &= \mathcal{F}\left[\cos\left(\frac{k}{\sqrt{6}}[\cos\alpha\ (x+y-2z)-\sqrt{3}\sin\alpha\ (x-y)]\right)\right] \\
&= \frac{e^{-2i\frac{k}{\sqrt{6}}z\cos\alpha}}{2}\ \delta\left(\frac{k}{\sqrt{6}}(\cos\alpha-\sqrt{3}\sin\alpha)-q_x\right)\delta\left(\frac{k}{\sqrt{6}}(\cos\alpha+\sqrt{3}\sin\alpha)-q_y\right) \\
&+\frac{e^{2i\frac{k}{\sqrt{6}}z\cos\alpha}}{2}\ \delta\left(\frac{k}{\sqrt{6}}(\cos\alpha-\sqrt{3}\sin\alpha)+q_x\right)\delta\left(\frac{k}{\sqrt{6}}(\cos\alpha+\sqrt{3}\sin\alpha)+q_y\right)
\end{aligned} \tag{19}$$

Then we can again compute I, II and III separately, introducing:

$$\gamma=\sqrt{\cos^2\alpha+3\sin^2\alpha} \tag{20}$$

$$\begin{aligned}
I &= \iint_{-\infty}^{\infty} iq_x\mathcal{F}(M_x)\ \mathcal{F}\left(\frac{1}{r}\right)\ e^{ixq_x}e^{iyq_y}\,\mathrm{d}q_x\,\mathrm{d}q_y \\
I &= \iint_{-\infty}^{\infty} -iq_x\frac{m_{\mathrm{DM}}}{\sqrt{6}}(\sqrt{3}\cos\alpha+\sin\alpha)\ 2\pi\frac{e^{-|q|(z-z')}}{|q|}\ \mathcal{F}\left[\cos\left(\vec{k_3'}(\alpha)\cdot\vec{r}\right)\right]e^{ixq_x}e^{iyq_y}\,\mathrm{d}q_x\,\mathrm{d}q_y \\
I &= \frac{m_{\mathrm{DM}}}{\sqrt{6}}(\sqrt{3}\cos\alpha+\sin\alpha)\ 2\pi\frac{e^{-\frac{\gamma k}{\sqrt{6}}(z-z')}}{\frac{\gamma k}{\sqrt{6}}}\frac{k}{\sqrt{6}}(\cos\alpha-\sqrt{3}\sin\alpha) \\
&\quad\times\frac{1}{2i}\left[e^{-2i\frac{k}{\sqrt{6}}z\cos\alpha}e^{i\frac{k}{\sqrt{6}}x(\cos\alpha-\sqrt{3}\sin\alpha)}e^{i\frac{k}{\sqrt{6}}y(\cos\alpha+\sqrt{3}\sin\alpha)}\right. \\
&\quad\left.-\ e^{2i\frac{k}{\sqrt{6}}z\cos\alpha}e^{-i\frac{k}{\sqrt{6}}x(\cos\alpha-\sqrt{3}\sin\alpha)}e^{-i\frac{k}{\sqrt{6}}y(\cos\alpha+\sqrt{3}\sin\alpha)}\right] \\
I &= \frac{2\pi m_{\mathrm{DM}}}{\gamma\sqrt{6}}\sqrt{3}(\cos^2\alpha-\sin^2\alpha)\ e^{-\frac{\gamma k}{\sqrt{6}}(z-z')}\sin\left(\vec{k_3'}(\alpha)\cdot\vec{r}\right)
\end{aligned} \tag{21}$$

$$\begin{aligned}
II &= \iint_{-\infty}^{\infty} iq_y\mathcal{F}(M_y)\ \mathcal{F}\left(\frac{1}{r}\right)\ e^{ixq_x}e^{iyq_y}\,\mathrm{d}q_x\,\mathrm{d}q_y \\
II &= \iint_{-\infty}^{\infty} iq_y\frac{m_{\mathrm{DM}}}{\sqrt{6}}(\sqrt{3}\cos\alpha-\sin\alpha)\ 2\pi\frac{e^{-|q|(z-z')}}{|q|}\ \mathcal{F}\left[\cos\left(\vec{k_3'}(\alpha)\cdot\vec{r}\right)\right]e^{ixq_x}e^{iyq_y}\,\mathrm{d}q_x\,\mathrm{d}q_y \\
II &= -\frac{m_{\mathrm{DM}}}{\sqrt{6}}(\sqrt{3}\cos\alpha-\sin\alpha)\ 2\pi\frac{e^{-\frac{\gamma k}{\sqrt{6}}(z-z')}}{\frac{\gamma k}{\sqrt{6}}}\frac{k}{\sqrt{6}}(\cos\alpha+\sqrt{3}\sin\alpha) \\
&\quad\times\frac{1}{2i}\left[e^{-2i\frac{k}{\sqrt{6}}z\cos\alpha}e^{i\frac{k}{\sqrt{6}}x(\cos\alpha-\sqrt{3}\sin\alpha)}e^{i\frac{k}{\sqrt{6}}y(\cos\alpha+\sqrt{3}\sin\alpha)}\right. \\
&\quad\left.-\ e^{2i\frac{k}{\sqrt{6}}z\cos\alpha}e^{-i\frac{k}{\sqrt{6}}x(\cos\alpha-\sqrt{3}\sin\alpha)}e^{-i\frac{k}{\sqrt{6}}y(\cos\alpha+\sqrt{3}\sin\alpha)}\right] \\
II &= -\frac{2\pi m_{\mathrm{DM}}}{\gamma\sqrt{6}}\sqrt{3}(\cos^2\alpha-\sin^2\alpha)\ e^{-\frac{\gamma k}{\sqrt{6}}(z-z')}\sin\left(\vec{k_3'}(\alpha)\cdot\vec{r}\right)=-I
\end{aligned} \tag{22}$$

$$
\begin{aligned}
III &= \iint_{-\infty}^{\infty} -|q|\ \mathcal{F}(M_z)\mathcal{F}\left(\frac{1}{r}\right)\ e^{ixq_x}e^{iyq_y}\,\mathrm{d}q_x\,\mathrm{d}q_y \\
III &= \iint_{-\infty}^{\infty} -|q|\frac{2\sin\alpha\ m_{\mathrm{DM}}}{\sqrt{6}}\ 2\pi\frac{e^{-|q|(z-z')}}{|q|}\ \mathcal{F}\left[\cos\Big(\vec{k_3'}(\alpha)\cdot\vec{r}\Big)\right] e^{ixq_x}e^{iyq_y}\,\mathrm{d}q_x\,\mathrm{d}q_y \\
III &= -\frac{4\pi\sin\alpha\ m_{\mathrm{DM}}}{2\sqrt{6}}e^{-\frac{\gamma k}{\sqrt{6}}(z-z')}\Big[e^{-2i\frac{k}{\sqrt{6}}z\cos\alpha}e^{i\frac{k}{\sqrt{6}}x(\cos\alpha-\sqrt{3}\sin\alpha)}e^{i\frac{k}{\sqrt{6}}y(\cos\alpha+\sqrt{3}\sin\alpha)} \\
&\qquad\qquad - e^{2i\frac{k}{\sqrt{6}}z\cos\alpha}e^{-i\frac{k}{\sqrt{6}}x(\cos\alpha-\sqrt{3}\sin\alpha)}e^{-i\frac{k}{\sqrt{6}}y(\cos\alpha+\sqrt{3}\sin\alpha)}\Big] \\
III &= -\frac{4\pi\sin\alpha\ m_{\mathrm{DM}}}{\sqrt{6}}e^{-\frac{\gamma k}{\sqrt{6}}(z-z')}\cos\Big(\vec{k_3'}(\alpha)\cdot\vec{r}\Big)
\end{aligned}
\tag{23}
$$

Bringing everything together, we get the expression of Φ for a single BFO layer:

$$
\boxed{\Phi = \frac{2\mu_0 m_{\mathrm{DM}}\sin\alpha\ e^{-\frac{\gamma k z}{\sqrt{6}}}}{k\gamma a^3}\sinh\left(\frac{\gamma k a}{2\sqrt{6}}\right)\cos\Big(\vec{k_3'}(\alpha)\cdot\vec{r}\Big)}
\tag{24}
$$

and the corresponding stray field:

$$
\begin{cases}
B_x^{\text{one layer}} = \dfrac{2\mu_0 m_{\mathrm{DM}}}{\gamma\sqrt{6}\ a^3}\ \sin\alpha\ (\cos\alpha-\sqrt{3}\sin\alpha)\ \sinh\left(\dfrac{k\gamma a}{2\sqrt{6}}\right)\ e^{-\frac{k\gamma z}{\sqrt{6}}}\ \sin\Big(\vec{k_3'}(\alpha)\cdot\vec{r}\Big) \\
B_y^{\text{one layer}} = \dfrac{2\mu_0 m_{\mathrm{DM}}}{\gamma\sqrt{6}\ a^3}\ \sin\alpha\ (\cos\alpha+\sqrt{3}\sin\alpha)\ \sinh\left(\dfrac{k\gamma a}{2\sqrt{6}}\right)\ e^{-\frac{k\gamma z}{\sqrt{6}}}\ \sin\Big(\vec{k_3'}(\alpha)\cdot\vec{r}\Big) \\
B_z^{\text{one layer}} = -\dfrac{2\mu_0 m_{\mathrm{DM}}}{\sqrt{6}\ a^3}\sin\alpha\ \sinh\left(\dfrac{k\gamma a}{2\sqrt{6}}\right)\ e^{-\frac{k\gamma z}{\sqrt{6}}}\left[-\cos\Big(\vec{k_3'}(\alpha)\cdot\vec{r}\Big)+\dfrac{2\cos\alpha}{\gamma}\sin\Big(\vec{k_3'}(\alpha)\cdot\vec{r}\Big)\right]
\end{cases}
\tag{25}
$$

To get the value of the field for the whole film constituted of N layers ($t = aN$), we have to sum these terms with $z \to z + ja$ ($j \in [0, N-1]$). At $z = d_{\mathrm{NV}}$, this leads to Eq.(13), (14) and (15) of the main text:

$$
\boxed{\begin{cases}
B_x = \dfrac{\mathcal{A}}{\gamma}\ (\cos\alpha-\sqrt{3}\sin\alpha)\ \operatorname{Im}\{\mathcal{S}\} \\
B_y = \dfrac{\mathcal{A}}{\gamma}\ (\cos\alpha+\sqrt{3}\sin\alpha)\ \operatorname{Im}\{\mathcal{S}\} \\
B_z = \mathcal{A}\left(\operatorname{Re}\{\mathcal{S}\}-\dfrac{2\cos\alpha}{\gamma}\operatorname{Im}\{\mathcal{S}\}\right)
\end{cases}}
\tag{26}
$$

where

$$
\mathcal{A} = \frac{2\mu_0 m_{\mathrm{DM}}}{\sqrt{6}\ a^3}\ \sin\alpha\ \sinh\left(\frac{\gamma k a}{2\sqrt{6}}\right)e^{-\frac{\gamma k d_{\mathrm{NV}}}{\sqrt{6}}}
\tag{27}
$$

and

$$
\mathcal{S} = e^{i\vec{k_3'}(\alpha)\cdot\vec{r}}\ \frac{1-e^{-\frac{kt}{\sqrt{6}}(\gamma+2i\cos\alpha)}}{1-e^{-\frac{ka}{\sqrt{6}}(\gamma+2i\cos\alpha)}}
\tag{28}
$$

II. ESTIMATION OF THE ERROR ON m_{DM}

The uncertainty on the value of m_{DM} arises from the uncertainty of the fitting procedure, from the uncertainty on the measurement of B_{NV} and from the uncertainties on the parameters θ, ϕ, d_{NV}, t and α. We follow the procedure described in *Gross* et al, *Nature 549, 252 (2017)* to derive the uncertainty on m_{DM}. The parameters p_i are expressed as $p_i = \bar{p_i} \pm \sigma_{p_i}$, where $\bar{p_i}$ is the nominal value and σ_{p_i} the standard error on each parameter. To estimate the relative uncertainty ε_{p_i} introduced by each parameter, we perform the fit with one parameter p_j fixed at $\bar{p_j} \pm \sigma_{p_j}$ and the others at their nominal value, and we define ε_{p_j} as:

$$\varepsilon_{p_j} = \frac{m_{\mathrm{DM}}(\bar{p_j} + \sigma_{p_j}) - m_{\mathrm{DM}}(\bar{p_j} + \sigma_{p_j})}{2 m_{\mathrm{DM}}(\bar{p_j})} \tag{29}$$

Once this is done for every parameter, we calculate the cumulative uncertainty ε:

$$\varepsilon = \sqrt{\varepsilon_{\mathrm{fit}} + \varepsilon_{\mathrm{meas}} + \sum_i \varepsilon_{p_i}} \tag{30}$$

The different values associated to each parameter are gathered in Table I. The main source of uncertainty on m_{DM} is the error on the NV-to-sample distance d_{NV}.

	Type I cycloid			Type II cycloid		
Parameter	Nominal value	Uncertainty	ε_{p_i}	Nominal value	Uncertainty	ε_{p_i}
θ	60.1°	0.5°	0.3%	60.1°	0.5°	0.3%
ϕ	−90.2°	1°	1%	−90.2°	1°	1%
d_{NV}	62.7 nm	2.2 nm	23%	62.7 nm	2.2 nm	8%
t	30 nm	2 nm	1%	54 nm	2 nm	1%
α	-	-	-	4.5°	1°	20 %
Meas. B_{NV}		10 µT	12%		10 µT	8%
Fit			4%			5%
Total			$\varepsilon = 26\%$			$\varepsilon = 24\%$

TABLE I. Table gathering the different uncertainties contributing to the uncertainty on m_{DM}.